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Tile Digit Sets of Integral Self-affine Tilings

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Abstract: Integral self-affine tilings generated by an expanding integer matrix $A \in M_n(\mathbb{Z})$ and $\mathcal{D} = \{d_1, d_2, \dots, d_m\} \subseteq \mathbb{Z}^n$ have been studied by many works. An important problem is to decide when a digit set gives us a tile (and then we call it a tile digit set). It is shown that the standard digit sets by Bandt, product form digit sets by Lagarias and Wang, and weak-product form digit sets in \mathbb{R}^1 by Lau and Rao are tile digit sets. In this paper, we generalize the notion of weak product form to higher dimensions and prove that they are tile digit sets.

Keywords: integral self-affine tiling; standard digit set; product form digit set; weak-product form digit set

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1 Introduction

Let A be an expanding integer matrix in $M_n(\mathbb{Z})$ (all eigenvalues $\lambda_i(A) > 1$). Denote $m = |\det(A)|$. Let $\mathcal{D} = \{d_1, d_2, \dots, d_m\} \subseteq \mathbb{Z}^n$ be a finite set of vectors, called digit set. Then $\phi_i(x) = A^{-1}(x + d_i)$, $1 \leq i \leq m$ are all contractions^[1]. There is a unique compact set $T = T(A, \mathcal{D})$ satisfying^[2]

$$T = \bigcup_{i=1}^m \phi_i(T). \quad (1)$$

T is given explicitly by $T := \left\{ \sum_{k=1}^{\infty} A^{-k} d_k : d_k \in \mathcal{D} \right\}$. An equivalent form of the functional equation (1) is $A(T) = \bigcup_{i=1}^m (T + d_i)$. When the attractor $T(A, \mathcal{D})$ has a positive Lebesgue measure, we call it an integral self-affine tile, and \mathcal{D} a tile digit set. In this case, it is well known that such T tiles \mathbb{R}^n by some translation set $\mathcal{S} \subseteq \mathbb{Z}^n$ (see [1]).

First, let us introduce some notations. $\mathcal{D}_{A,k} = \left\{ \sum_{j=0}^{k-1} A^j d_{i_j} : \text{all } d_{i_j} \in \mathcal{D} \right\}$, $\mathcal{D}_{A,\infty} = \bigcup_{k=1}^{\infty} \mathcal{D}_{A,k}$. Note that $0 \in \mathcal{D}$ implies that $\mathcal{D}_{A,k} \subseteq \mathcal{D}_{A,k+1}$ for all $k \geq 1$. We say that a set $\mathcal{V} \subseteq \mathbb{R}^n$ is uniformly discrete if there exists $\delta > 0$ such that $v, v' \in \mathcal{V}$ implies $|v - v'| > \delta$, where $|\cdot|$ is the Euclidean norm in \mathbb{R}^n .

The following theorem gives criterion for $T(A, \mathcal{D})$ being a self-affine tile^[1].

Theorem 1.1 (Interior Theorem) Let $A \in M_n(\mathbb{R})$ be an expanding matrix such that $|\det(A)| = m$ is an integer. Let $\mathcal{D} \subseteq \mathbb{R}^n$ have cardinality m , and suppose that $0 \in \mathcal{D}$. The following four conditions are equivalent.

(i) $T(A, \mathcal{D})$ has positive Lebesgue measure;

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- (ii) $T(\mathbb{A}, \mathcal{D})$ has nonempty interior;
- (iii) $T(\mathbb{A}, \mathcal{D})$ is the closure of its interior T° , and its boundary $\partial T := T - T^\circ$ has Lebesgue measure zero;

(iv) For each $k \geq 1$, $\mathcal{D}_{\mathbb{A},k}$ has m^k distinct elements and $\mathcal{D}_{\mathbb{A},\infty}$ is a uniformly discrete set.

If \mathcal{D} is a standard digit set, then it is a tile digit set^[3]. It is introduced in [4] digit sets of product form and showed they are tile digit set. In the 1-dimension case, it is defined in [5] a class of weak product forms and showed that they are tile digit sets. In this paper, we generalize the result of [5] to higher dimensions.

2 Weak product-form digit sets

Recall that a digit set $(\mathbb{A}, \mathcal{D})$ is a product-form digit set if \mathcal{D} has an additive factorization

$$\mathcal{D} = \mathbb{A}^{f(1)}(\mathcal{E}_1) + \mathbb{A}^{f(2)}(\mathcal{E}_2) + \cdots + \mathbb{A}^{f(r)}(\mathcal{E}_r),$$

in which $r \geq 2$, and $0 \leq f(1) \leq f(2) \leq \cdots \leq f(r)$, and where $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_r \subseteq \mathbb{Z}^n$, each have $0 \in \mathcal{E}_i$, and $\mathcal{E} := \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_r$ is a complete set of coset representatives of $\mathbb{Z}^n / \mathbb{A}(\mathbb{Z}^n)$, that is $\#(\mathcal{E}) = \#(\mathcal{E}_1)\#(\mathcal{E}_2) \cdots \#(\mathcal{E}_r) = |\det(\mathbb{A})|$. If some $f(i) > 0$, then this is a nonstandard digit set and we have the following theorem^[4].

Theorem 2.1 A product-form digit set tile $T(\mathbb{A}, \mathcal{D})$ is a measure-disjoint union of translates of $T(\mathbb{A}, \mathcal{E})$, and has measure

$$\mu(T(\mathbb{A}, \mathcal{D})) = \mu(T(\mathbb{A}, \mathcal{E})) \prod_{i=1}^r \#(\mathcal{E}_i)^{f(i)}. \quad (2)$$

It is showed in [4] that, for the above \mathcal{D} and \mathcal{E} , there exists $\mathcal{W} \subseteq \mathbb{Z}^n$ such that

$$\mathbb{A}^{f(r)}T(\mathbb{A}, \mathcal{E}) = T(\mathbb{A}, \mathcal{D}) + \mathcal{W}. \quad (3)$$

Since $(T(\mathbb{A}, \mathcal{E}), \mathbb{Z}^n)$ is a tiling of \mathbb{R}^n , it follows that $(\mathbb{A}^{f(r)}T(\mathbb{A}, \mathcal{E}), \mathbb{A}^{f(r)}\mathbb{Z}^n)$ is also a tiling of \mathbb{R}^n . Therefore $(T(\mathbb{A}, \mathcal{D}), \mathcal{W} + \mathbb{A}^{f(r)}\mathbb{Z}^n)$ is a tiling of \mathbb{R}^n . We denote that

$$\mathcal{J} = \mathcal{W} + \mathbb{A}^{f(r)}\mathbb{Z}^n. \quad (4)$$

We need the following simple result in [3].

Lemma 2.1^[6] If $\mathbb{A}^p T(\mathbb{A}, \mathcal{D})$ can be tiled by $T(\mathbb{A}, \mathcal{D})$ with two translation sets \mathcal{J} and \mathcal{J}' where $p \in \mathbb{Z}$, then $\mathcal{J} = \mathcal{J}'$.

Theorem 2.2 Let $\mathcal{D} \subseteq \mathbb{Z}^n$ be a product-form digit set, and $\mathcal{J} = \mathcal{W} + \mathbb{A}^{f(r)}\mathbb{Z}^n$. Then $\mathcal{J} = \mathbb{A}\mathcal{J} + \mathcal{D}$.

Proof Since

$$\begin{aligned} \mathbb{A}T(\mathbb{A}, \mathcal{E}) &= \mathbb{A} \left\{ \sum_{k=1}^{\infty} \mathbb{A}^{-k} e_k : e_k \in \mathcal{E} \right\} = \left\{ \sum_{k=0}^{\infty} \mathbb{A}^{-k} e_k : e_k \in \mathcal{E} \right\} \\ &= \left\{ e_0 + \sum_{k=1}^{\infty} \mathbb{A}^{-k} e_k : e_0, e_k \in \mathcal{E} \right\} = \mathcal{E} + T(\mathbb{A}, \mathcal{E}), \end{aligned}$$

$$\mathbb{A}^{f(r)+1}T(\mathbb{A}, \mathcal{E}) = \mathbb{A}^{f(r)}(\mathcal{E} + T(\mathbb{A}, \mathcal{E})) = \mathbb{A}^{f(r)}\mathcal{E} + \mathcal{W} + T(\mathbb{A}, \mathcal{D}).$$

And

$$\mathbb{A}^{f(r)+1}T(\mathbb{A}, \mathcal{E}) = \mathbb{A}(\mathcal{W} + T(\mathbb{A}, \mathcal{D})) = \mathbb{A}\mathcal{W} + \mathcal{D} + T(\mathbb{A}, \mathcal{D}).$$

Then by Lemma 2.1, $\mathbb{A}^{f(r)}\mathcal{E} + \mathcal{W} = \mathbb{A}\mathcal{W} + \mathcal{D}$. Hence

$$\begin{aligned}\mathbb{A}\mathcal{J} + \mathcal{D} &= \mathbb{A}\mathcal{W} + \mathbb{A}^{f(r)+1}\mathbb{Z}^n + \mathcal{D} = \mathbb{A}^{f(r)}\mathcal{E} + \mathcal{W} + \mathbb{A}^{f(r)+1}\mathbb{Z}^n \\ &= \mathbb{A}^{f(r)}(\mathcal{E} + \mathbb{A}\mathbb{Z}^n) + \mathcal{W} = \mathbb{A}^{f(r)}\mathbb{Z}^n + \mathcal{W} = \mathcal{J}.\end{aligned}$$

We generalize the notion of weak product form in [5] to \mathbb{R}^n .

Definition 2.1 A digit set $\mathcal{D} \subseteq \mathbb{Z}^n$ is called a weak product-form digit set if there is a product-form digit set \mathcal{D}' with

$$\mathcal{D}' = \mathbb{A}^{f(1)}(\mathcal{E}_1) + \mathbb{A}^{f(2)}(\mathcal{E}_2) + \cdots + \mathbb{A}^{f(r)}(\mathcal{E}_r),$$

such that $\mathcal{D} \equiv \mathcal{D}' \pmod{\mathbb{A}^{f(r)+1}}$.

Definition 2.2 We say a set $\mathcal{J} \subseteq \mathbb{Z}^n$ has positive density in \mathbb{Z}^n , if

$$\overline{\lim}_{m \rightarrow \infty} \frac{1}{(2m)^n} \#\{t \in \mathcal{J} : |t| \leq m\} > 0,$$

where for $t = (t_1, t_2, \dots, t_n)$, we define $|t|_\infty = \max\{t_1, t_2, \dots, t_n\}$.

Lemma 2.2 Let $\mathcal{D} \subseteq \mathbb{Z}^n$ be a digit set. Suppose there is a set $\mathcal{J} \subseteq \mathbb{Z}^n$ such that

- (i) $\mathcal{J} \subseteq \mathbb{A}\mathcal{J} + \mathcal{D}$;
- (ii) \mathcal{J} has positive density in \mathbb{Z}^n .

Then \mathcal{D} is a tile digit set.

Proof It is well known that \mathcal{D} is a tile digit set if and only if $\#(\mathcal{D}_{\mathbb{A},k}) = b^k$. Hence if \mathcal{D} is not a tile digit set, there exists a $k \geq 1$ such that $\#(\mathcal{D}_{\mathbb{A},k}) < b^k$. From

$$\begin{aligned}\mathcal{D}_{\mathbb{A},km} &= \left\{ \sum_{j=0}^{km-1} \mathbb{A}^j d_j : d_j \in \mathcal{D} \right\} \\ &= \mathbb{A}^{(m-1)k} \mathcal{D}_{\mathbb{A},k} + \cdots + \mathbb{A} \mathcal{D}_{\mathbb{A},k} + \mathcal{D}_{\mathbb{A},k} = (\mathcal{D}_{\mathbb{A},k})_m,\end{aligned}$$

we deduce that $\# \mathcal{D}_{\mathbb{A},km} \leq (\# \mathcal{D}_{\mathbb{A},k})^m$. Therefore

$$\lim_{m \rightarrow \infty} \frac{\# \mathcal{D}_{\mathbb{A},km}}{b^{km}} \leq \lim_{n \rightarrow \infty} \frac{(\# \mathcal{D}_{\mathbb{A},k})^m}{b^{km}} = \lim_{m \rightarrow \infty} \left(\frac{\# \mathcal{D}_{\mathbb{A},k}}{b^k} \right)^m \leq \lim_{m \rightarrow \infty} \left(\frac{b^k - 1}{b^k} \right)^m = 0,$$

$$\frac{\# \mathcal{D}_{\mathbb{A},m+1}}{b^{m+1}} = \frac{\#(\mathcal{D}_{\mathbb{A},m} + \mathcal{D})}{b^{m+1}} \leq \frac{\# \mathcal{D}_{\mathbb{A},m}}{b^m} \cdot \frac{\# \mathcal{D}}{b} = \frac{\# \mathcal{D}_{\mathbb{A},m}}{b^m},$$

which means that $\frac{\# \mathcal{D}_m}{b^m}$ is non-increasing on n . This yields that

$$\lim_{m \rightarrow \infty} \frac{\# \mathcal{D}_m}{b^m} = 0.$$

Now, if we repeat the inclusion in (i) for k times, we have

$$\mathcal{J} \subseteq \mathbb{A}^k \mathcal{J} + \mathcal{D}_k \subseteq \mathbb{A}^k \mathbb{Z}^n + \mathcal{D}_k.$$

So the density of \mathcal{J} is not larger than the density of $\mathbb{A}^k \mathbb{Z}^n + \mathcal{D}_k$, which is at most $\# \frac{\mathcal{D}_k}{b^k}$. We see from this that the density of \mathcal{J} is 0. This is a contradiction; so \mathcal{D} must be a tile digit set.

Theorem 2.3 If \mathcal{D} is a weak product-form digit set, then \mathcal{D} is a tile digit set.

Proof Let \mathcal{D}' be the associated product-form digit set as in the definition and $\mathcal{J} = \mathcal{W} + \mathbb{A}^{f(r)} \mathbb{Z}^n$ be the translation set as in Theorem 2.2. Clearly \mathcal{J} has positive density in \mathbb{Z}^n . On the other side, there is a $t \in \mathbb{Z}^n$ such that $\mathcal{D} = \mathcal{D}' + (\mathbb{A}^{f(r)+1})t$, so

$$\begin{aligned} \mathbb{A}\mathcal{J} + \mathcal{D} &= \mathbb{A}\mathcal{W} + \mathbb{A}^{f(r)+1} \mathbb{Z}^n + \mathcal{D}' + \mathbb{A}^{f(r)+1} t \\ &= \mathbb{A}^{f(r)} \mathcal{E} + \mathcal{W} + \mathbb{A}^{f(r)+1} \mathbb{Z}^n + \mathbb{A}^{f(r)+1} t \\ &= \mathbb{A}^{f(r)} (\mathcal{E} + \mathbb{A} \mathbb{Z}^n + \mathbb{A} t) + \mathcal{W} = \mathbb{A}^{f(r)} \mathbb{Z}^n + \mathcal{W} = \mathcal{J}. \end{aligned}$$

So \mathcal{J} satisfies the two conditions of Lemma 2.2. Therefore \mathcal{D} is a tile digit set.

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整数自仿 Tiling 的 Tile 数字集

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摘要: 关于由一个扩张矩阵 $\mathbb{A} \in M_n(\mathbb{Z})$ 和数字集 $\mathcal{D} = \{d_1, d_2, \dots, d_m\} \subseteq \mathbb{Z}^n$ 生成的整数自仿 Tiling, 已经有很多研究结果。其中一个重要的问题是判定一个数字集在什么条件下能生成一个 Tile。在一维情况下, 已知结果有: 标准数字集, 乘积形式数字集, 弱乘积形式数字集都是 Tile 数字集。在本文中, 我们把弱乘积形式的概念推广到高维, 并证明它们都是 Tile 数字集。

关键词: 整数自仿 Tile; 标准数字集; 乘积形式数字集; 弱乘积形式数字集